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A general solution of the Hele–Shaw problem for flows in a channel☆

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Abstract

The partial derivatives of the function which maps the auxiliary plane into the physical plane are rational functions for all known exact solutions of the problem of fingering in a Hele–Shaw cell. Using methods of complex analysis a general form of the solution is constructed which possesses this property and, unlike existing solutions, is not necessarily symmetrical about the central axis of the cell. The generalized problem of the dynamics of the system is written for the free parameters of the solution and it is shown to be completely integrable.

The classical Hele–Shaw problem describes the evolution of the interface between a non-viscous and a viscous liquid (henceforth air and simply a liquid) in their combined flow in a Hele–Shaw cell, assuming continuity of the pressure at the interface.[1] When the liquid retreats, the interface becomes unstable. Nevertheless, in experiments on cells of different geometry[1,2] relatively stable structures of the growing "fingers" type are observed.[1] This is a characteristic feature of numerous processes, mathematical models of which reduce to a Hele–Shaw problem,[3] and also to different generalizations of it, for example, related to the non-Newtonian rheology of a viscous liquid.[4]

The first exact solutions of the problem of the constriction of the contour of oil-bearing stratum, related to the Hele–Shaw problem, were obtained in the middle of the last century.[5–7] The first non-stationary solution was constructed[8] just for the problem of fingering in a Hele–Shaw cell. For the problem of an "inflating bubble" it was shown for the first time in Ref. 9 that the Hele–Shaw boundary-value problem can be reduced to a generalized system-dynamics problem. It was suggested that the first integrals of such a dynamical system could be found using the Schwarz function, and in fact a method of constructing exact solutions (henceforth called the Howison method) was formulated.[10,11] Using this method a set of exact solutions of both Hele–Shaw flows and processes related to them was constructed.[12] However, the method itself is not well formalized, since only the idea of possible ways of generalizing the particular solutions obtained were indicated.[11] Moreover, the method is not complete – by itself it does not enable one to assert categorically whether the parametrized form of the mapping function is a solution of the Hele–Shaw problem or not. In addition to the introduction of an auxiliary plane of the complex variable,[5,6] a common element of all the publications mentioned is the representation of the solution in the form of a parametrized function, which maps the auxiliary plane into the physical plane. All the solutions obtained are special representatives of a class of solutions in which the partial derivatives of the mapping function are rational functions in the auxiliary plane[11] (there are solutions which

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do not fall into this class,[13] but only for flows in cells with a corner geometry). The purpose of this paper is to construct an exact solution of the most general form of this class of solutions of the Hele–Shaw problem for flows in a channel.

1. The governing relations for Hele–Shaw flows

The mathematical formulation of the classical Hele–Shaw problem has the form[1]

$$\Omega_z(t)\colon \Delta p = 0; \quad \Gamma(t)\colon -\partial p/\partial n = \upsilon_n, \quad p = 0 \tag{1.1}$$

to which we must add the natural boundary conditions on the physical boundaries of the cell and at infinity. Here $\Omega_z(t)$ is the region occupied by the liquid, $\Gamma(t)$ is the interface between the liquid and air, n is the outward normal to $\Gamma(t)$, $p(x, y, t)$ is the pressure in the liquid and υ_n is the normal component of the velocity of motion of the interface. The first boundary condition in problem (1.1) is the kinematic boundary condition and the second is the dynamic boundary condition and in this form is often called the idealized boundary condition.

Boundary-value problem (1.1) enables us to introduce the complex physical plane $z = x + iy$ and the complex flow potential $W = \varphi + i\psi$, where $\varphi = -p$ and ψ is the stream function, where $W = W(z, t)$.[14] The use of complex variables also makes it more convenient to introduce complex-valued analogues of the vector fields.[15] If a vector field $\mathbf{V}(x, y, t) = (\upsilon_x, \upsilon_y)$ acts in the $\mathbf{R}^2$ plane, we can then obviously speak of a vector field $\mathbb{V}(z, \bar{z}, t)$

$$\mathbb{V}(z, \bar{z}, t) = [\upsilon_x(x, y, t) + i\upsilon_y(x, y, t)]|_{x = (z+\bar{z})/2;\ y = (z-\bar{z})/2} \tag{1.2}$$

which acts in the complex plane $\mathbb{C}$. We can use the formula $\mathbb{V}_p = \overline{\partial W/\partial z}$ for the velocity of potential flow, whence, in particular, it follows that $\mathbb{V}_p = \mathbb{V}_p(\bar{z}, t)$. If the form of the function $\mathbb{V}_p(\bar{z}, t)$ is parametrized, we can write $\partial W/\partial z = \mathbb{V}_p^*(z, t)$, where $\mathbb{V}_p^*(z, t) \equiv \overline{\mathbb{V}_p(\bar{z}, t)}$. Here and henceforth the asterisk denotes the operation of conjugation only with respect to the parameters of a function (and not the variable).

We will introduce an auxiliary plane of the complex variable ζ. The region Ω_ζ of unchanged form corresponds to the region $\Omega_z(t)$ in it. The actual form of the region is not yet specified, but we agree henceforth to choose it so that the free boundary Γ in the ζ plane corresponds to the arc of the unit circle. We will denote the analytical function, which conformally maps the region Ω_ζ into the region $\Omega_z(t)$ by $g(\zeta, t)$:

$$z = g(\zeta, t), \quad |\partial g/\partial \zeta|_{\zeta \in \Omega_\zeta} \neq 0, \infty \tag{1.3}$$

The conformality of the mapping presupposes that there are no singularities of the function $g(\zeta, t)$ in the region Ω_ζ, which is stressed in expression (1.3). As a consequence the inverse mapping

$$\zeta = f(z, t), \quad |\partial f/\partial z|_{z \in \Omega_z} \neq 0, \infty \tag{1.4}$$

also exists.

The conformal mapping parameter (1.3), which depends on the time t, specifies a group of transformations to which a certain conformal motion of points of the physical plane corresponds. The auxiliary plane ζ in this case emerges as the plane of Lagrange variables. We will denote the velocity of this motion by $\mathbb{V}_g = (z, t)$ and, by definition, we have

$$\mathbb{V}_g(z, t) = (\partial g/\partial t)|_{\zeta = f(z, t)} \tag{1.5}$$

The different forms of the potential motion $\mathbb{V}_p = (\bar{z}, t)$ and the conformal motion $\mathbb{V}_g = (z, t)$ may lead to the same evolution of the free boundary if and only if at each point of the boundary the normal projection of the velocities of these motions coincide:

$$\forall t, z \in \Gamma(t)\colon \mathrm{Re}\{\bar{\mathbb{V}}_g \mathbb{N}\} = \mathrm{Re}\{\bar{\mathbb{V}}_p \mathbb{N}\} \tag{1.6}$$

where $\mathbb{N}$ is the complex-valued analogue of the outward normal to the boundary $\Gamma(t)$ in the z plane, while for the scalar product of the two vectors $\mathbb{V}_1$ and $\mathbb{V}_2$ we have used the obvious formula $\mathbb{V}_1 \cdot \mathbb{V}_2 = Re\{\mathbb{V}_1 \mathbb{V}_2\}$.

Boundary condition (1.6) is, in fact, the vector analogue of the Polubarinova-Galin boundary equation.[5,6] In order to convince ourselves of this we note that ζ is the complex analogue of the normal to the boundary Γ of the region Ω_ζ. Then, taking into account the geometrical properties of the derivative of an analytic function of a complex variable,[16] we obtain for the normal $\mathbb{N}$ the expression $\mathbb{N} = -\zeta e^{i\arg(\partial g/\partial\zeta)}$. Substituting it into condition (1.6) and using representation (1.5) for the velocity $\mathbb{V}_g$ and $\overline{\partial W/\partial z}$ for the velocity $\mathbb{V}_p$, we obtain the Polubarinova-Galin equation of general form

$$\zeta = e^{i\sigma}\colon \operatorname{Re}\{(\overline{\partial g/\partial t})\zeta\partial g/\partial\zeta\} = \operatorname{Re}\{\zeta\partial W/\partial\zeta\} \tag{1.7}$$

where the right-hand side has been transformed taking into account the obvious differential relation

$$(\partial W/\partial z)(\partial g/\partial\zeta) = \partial W/\partial\zeta$$

Further, we will introduce the Schwarz function.[17] It is obtained by substituting

$$x = (z+\bar{z})/2, \quad y = (z-\bar{z})/2$$

into the equation $F(x, y, t) = 0$ of the moving free boundary and solving it for $\bar{z} : \bar{z} = S(z, t)$, where $z \in \Gamma(t)$.

We will denote the transform of the Schwarz function in the auxiliary plane by $r(\zeta, t) \equiv S(g(\zeta, t), t)$. In this plane the arc of the unit circle $\zeta = e^{i\sigma}$ corresponds to the free boundary $\Gamma(t)$. Correspondingly, the function $r(\zeta, t)$ is obtained by applying the following transformation[11] to the function $g(\zeta, t)$

$$r(\zeta, t) = g^*(\zeta^{-1}, t) \equiv \mathcal{P}[g(\zeta, t)] \tag{1.8}$$

which we will define as the operator $\mathcal{P}$. Using it we can also express the partial derivatives of the function $r(\zeta, t)$

$$\partial r/\partial t = \mathcal{P}[\partial g/\partial t], \quad \zeta(\partial r/\partial\zeta) = -\mathcal{P}[\zeta(\partial g/\partial\zeta)]$$

Since the following relations are satisfied on the free boundary

$$\zeta = e^{i\sigma}\colon \zeta^{-1} = \bar{\zeta}, \quad \partial r/\partial t = \overline{(\partial g/\partial t)}, \quad \zeta(\partial r/\partial\zeta) = -\overline{\zeta(\partial g/\partial\zeta)}$$

the boundary evolution equation (1.7) can be rewritten in the form

$$\zeta = e^{i\sigma}\colon \zeta\frac{\partial r}{\partial t}\frac{\partial g}{\partial\zeta} - \zeta\frac{\partial g}{\partial t}\frac{\partial r}{\partial\zeta} = 2\operatorname{Re}\left\{\zeta\frac{\partial W}{\partial\zeta}\right\} \tag{1.9}$$

It was precisely this form of the evolution equation that was taken in Ref. 7 to construct the solution of the problem of the contraction of the contour of an oil-bearing stratum (with the refinement that Kufarev did not use explicit notation for the Schwarz function), and was also employed in Ref. 18 to establish a relation between Hele–Shaw problems with the modern theory of integrable systems. The advantage of the form (1.9) over (1.7) is the fact that the left-hand side of the equation

$$\Phi(\zeta, t) \equiv \zeta\frac{\partial r}{\partial t}\frac{\partial g}{\partial\zeta} - \zeta\frac{\partial g}{\partial t}\frac{\partial r}{\partial\zeta} = \mathcal{P}\left[\frac{\partial g}{\partial t}\right]\zeta\frac{\partial g}{\partial\zeta} + \frac{\partial g}{\partial t}\mathcal{P}\left[\zeta\frac{\partial g}{\partial\zeta}\right] \tag{1.10}$$

is an analytic function of the variable ζ. This enables us to use the apparatus of the theory of functions of a complex variable to analyse it.

The second expression in (1.10) represents the function $\Phi(\zeta, t)$ by a certain transformation of the function $g(\zeta, t)$. From it, taking into account the obvious equivalence of the transformation $\mathcal{P}^2$ to an identity, it follows that the function $\Phi(\zeta, t)$ is invariant to the transformation $\mathcal{P}$, which enables us, for certain conditions, to establish the structure of the function.

Assertion. Suppose the partial derivatives of the functions $g(\zeta, t)$ with respect to the variables ζ and t are rational functions in the ζ plane. Then the function $\Phi(\zeta, t)$ is also rational and can be represented in the form

$$\Phi(\zeta, t) = \alpha(t)\prod_{j=1}^{J_n}[\zeta - c_j(t)][\zeta^{-1} - \bar{c}_j(t)]\Big/\prod_{j=1}^{J_p}[\zeta - b_j(t)][\zeta^{-1} - \bar{b}_j(t)]$$

where $\alpha(t)$ is a certain real function of t. In other words, the rational function $\Phi(\zeta, t)$ is a combination which differs from zero and infinity of the ζ plane of paired zeros $\{c_j, \bar{c}_j^{-1}\}$ and paired poles $\{b_j, \bar{b}_j^{-1}\}$.

The proof obviously follows from the multiplicative representation of a rational function and the invariance of $\Phi(\zeta, t)$ under the transformation $\mathcal{P}$. In this case the number of zeros $2J_n$ and poles $2J_p$ may not be the same. The sign of the

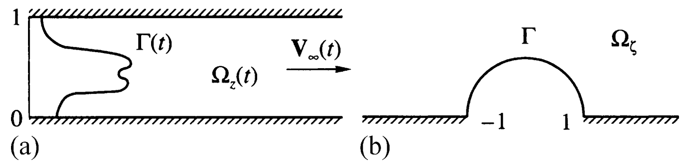

Fig. 1.

difference $J_n - J_p$ determines the behaviour of the function $\Phi(\zeta, t)$ at zero and infinity: if it is positive, there will be poles of order $J_n - J_p$ there, and if it is negative, there will be zeros of order $J_p - J_n$.

2. Flow in a channel

We will henceforth only consider flow in a Hele–Shaw cell, which has the shape of a channel with an idealized dynamic boundary condition on the free boundary. To system (1.1) we must add the conditions of impermeability of the side walls of the channel and specify, generally speaking, the non-constant rate of outflow of the liquid at infinity $V_\infty(t)$ (see Fig. 1a).

Specifying the form of the auxiliary ζ plane, we will choose as the region Ω_ζ the upper half of the exterior of the unit circle $|\zeta| = 1$ (Fig. 1b) so that the unit half-circle corresponds to the boundary $\Gamma(t)$ and the real axis corresponds to the channel walls while infinity corresponds to infinity

$$|\zeta| \sim \infty, \quad g(\zeta, t) \sim \pi^{-1}\ln\zeta; \quad \operatorname{Im}\zeta = 0, \quad \operatorname{Im}\{\partial g/\partial\zeta\} = 0 \tag{2.1}$$

The normalization of the conformal mapping (1.3) or, more accurately, two of the three necessary parameters of the normalization[16] are thereby specified. The latter parameter remains arbitrary until the solution is constructed.[10]

Further, taking into account the well-known results in Refs. 8–11, it is convenient to establish the structure of the solution of the problem, i.e. the most general parametric form of the function $g(\zeta, t)$, which ensures that its partial derivatives in the ζ plane are rational. In order to be able to extend the function $g(\zeta, t)$ to the real axis[16] when satisfying the normalization (2.1) we require that the solution should be symmetrical about the real axis in the auxiliary (but not in the physical) plane

$$g(\zeta, t) = \overline{g(\bar{\zeta}, t)} \equiv g^*(\zeta, t) \tag{2.2}$$

The general form of the derivative $\partial g/\partial\zeta$ as a rational function is as follows:

$$\pi\frac{\partial g}{\partial\zeta} = \zeta^{-M-1}\prod_{n=1}^{N}(\zeta - a_n)/\prod_{j=1}^{J}(\zeta - b_j), \quad N - J = M > 0 \tag{2.3}$$

We will denote by $a_n = a_n(t)$ and $b_j = b_j(t)$ respectively the zeros and the poles of the function, where the poles are assumed to be strictly different from the point $\zeta = 0$. A possible pole at this point is distinguished separately. The zeros $a_k(t)$ at any instant of time can coincide with the point $\zeta = 0$, so that after reducing the powers of ζ in the numerator and denominator in the function $\partial g/\partial\zeta$ at this point it may turn out to be in fact a zero of order no higher than $(J-1)$.

Like the singularities of the mapping function $g(\zeta, t)$, the zeros and poles of the derivative $\partial g/\partial\zeta$ must lie outside the region Ω_ζ.[16] More accurately, taking the symmetry of (2.2) into account, they must lie inside the unit circle $|\zeta| < 1$:

$$\forall t: |a_n| < 1, \quad n = 1, \ldots, N; \quad |b_j| < 1, \quad j = 1, \ldots, J \tag{2.4}$$

The rational function (2.3) also allows of an additive representation.[16] Integrating it with respect to ζ and choosing the constant of integration as fixed and equal to t, we obtain the general form of the function $g(\zeta, t)$

$$g(\zeta, t) = t + \frac{\ln\zeta}{\pi} + \sum_{j=1}^{J}\left[d_{j,0}\ln(\zeta^{-1} - b_j^{-1}) - \sum_{k=1}^{K_j}\frac{d_{j,k}}{(\zeta^{-1} - b_j^{-1})^k k}\right] + \sum_{m=1}^{M}\frac{\beta_m}{\zeta^m} \tag{2.5}$$

where $d_{j,0}$, $d_{j,k}$, b_j, β_m are free parameters, $d_{j,0}$ are complex constants, $d_{j,k}=d_{j,k}(t)$ and $b_j=b_j(t)$ are complex-valued quantities, and $\beta_m=\beta_m(t)$ are real functions of time. The zeros $a_n=a_n(t)$ of the derivative $\partial g/\partial\zeta$ will obviously be determined by the whole set of free parameters.

The realness of β_m is a consequence of the symmetry (2.2) of the function $g(\zeta, t)$. It necessarily leads to the condition of pairwise conjugation of all the remaining free parameters. In other words, the whole interval $j=1,\ldots, J$ of the singularities b_j can be conditionally split into three intervals

$$\begin{aligned}
&j\le J_0\colon \operatorname{Im}b_j = 0,\quad \operatorname{Im}d_{j,k} = 0,\quad \forall k = 1,\ldots,K_j\\
&J_0<j\le J_0+J_1\colon \operatorname{Im}b_j>0,\ b_j = \bar b_{j+J_1},\ d_{j,k} = \bar d_{j+J_1,k},\ \forall k = 1,\ldots,K_j\\
&j>J_0+J_1\colon \operatorname{Im}b_j<0,\ b_j = \bar b_{j-J_1},\ d_{j,k} = \bar d_{j-J_1,k},\ \forall k = 1,\ldots,K_j
\end{aligned}\tag{2.6}$$

Due to this pairwise conjugation only the following free parameters of the solution remain independent

$$[\beta_m(t), m = 1,\ldots,M];\quad [b_j(t), d_{j,k}(t), j = 1,\ldots,J_0+J_1, k = 1,\ldots,K_j]\tag{2.7}$$

Denoting the flow rate of the liquid in the cell by $Q(t)=V_\infty(t)$ and using the canonical form of the region Ω_ζ, we obtain the form of the function $W(\zeta, t)$

$$W(\zeta,t) = \pi^{-1}Q(t)\ln\zeta\tag{2.8}$$

Then Eq. (1.9) takes the form

$$\zeta = e^{i\sigma}\colon \zeta\frac{\partial r}{\partial t}\frac{\partial g}{\partial\zeta}-\zeta\frac{\partial g}{\partial t}\frac{\partial r}{\partial\zeta} = 2\frac{Q(t)}{\pi}\tag{2.9}$$

In accordance with the main idea of the method of Schwarz functions,[17] we will satisfy boundary equation (2.9) for a function $g(\zeta, t)$ of the form (2.5), thereby ensuring it is satisfied everywhere in the ζ plane. Taking into account the notation $\Phi(\zeta, t)$, introduced for the left-hand side of the equation, this is possible if we ensure that the equality $\Phi(\zeta, t)=2\pi^{-1}Q(t)$ is satisfied everywhere in the ζ plane.

It can be seen from expression (2.5) that both parts of the derivative of the function $g(\zeta, t)$ in the ζ plane are rational functions with poles at the zero and points b_j lying inside the unit circle. Consequently, the function $g(\zeta, t)$ satisfies all the conditions of the assertion from Section 1. Hence, the function $\Phi(\zeta, t)$ is also rational in the ζ plane and, moreover, consists structurally of two terms – rational functions with the same poles. Correspondingly, we can attempt to obtain a mutual reduction of the poles by finding additional conditions on the free parameters of the function $g(\zeta, t)$. Then, by Liouville's theorem[16] $\Phi(\zeta, t)$ in the whole ζ plane will be a function of only the parameter t, and will be real due to its invariance under the transformation $\mathcal{P}$. As a result, from the assertion derived in Section 1 for Hele–Shaw cell flows we have the following corollary.

Corollary. *Suppose the function $g(\zeta, t)$ has the form (2.5)–(2.6), while the function $\Phi(\zeta, t)$ is the transform of the function $g(\zeta, t)$ for tansformation (1.10). Then, the position of its zeros and poles will necessarily be symmetrical both relative to the unit circle and relative to the real axis in the ζ plane. If, by controlling the free parameters (2.7), we attempt to satisfy the set of local conditions in the neighbourhood of singular points of the function $\Phi(\zeta, t)$, lying in the closure of the region $\bar\Omega_\zeta$,*

$$\Phi(\zeta,t)|_{\zeta\sim\bar b_j^{-1}} = O(1),\quad j = 1,\ldots,J_0+J_1;\quad \Phi(\zeta,t)|_{|\zeta|\sim\infty} = O(1)\tag{2.10}$$

the function $g(\zeta, t)$ will be the solution of the Hele–Shaw problem for flow in a channel with a certain definite law of variation of the flow rate $Q=Q_*(t)$.

The defined form of the law of variation of the flow rate $Q=Q_*(t)$ is obviously related to the choice of a certain constant of integration of the function $g(\zeta, t)$ of the form (2.5). Henceforth by replacing the time variable[19] we can obtain a solution of the problem for any law $Q(t)>0$ specified in advance.

Knowing the form of the solution $g(\zeta, t)$ and the complex potential $W(\zeta)$, we can obtain the complex-conjugate velocity of the potential motion

$$\frac{\partial W}{\partial z} = \frac{Q(t)}{\pi\zeta(\partial g/\partial\zeta)} \tag{2.11}$$

Hence, when solution (2.5), (2.6) is parametrized, the Hele–Shaw boundary-value problem (1.1) for flows in a channel reduces to the problem of a local analysis of the behaviour of the function $\Phi(\zeta, t)$ in the neighbourhood of its singular point. Nevertheless, such a direct method involves cumbersome calculations, since the function itself will have a cumbersome form. We can eliminate these drawbacks by means of the formalism proposed below.

3. Formalism

Suppose we are given a continuous vector field $\mathbf{V}(x, y, t)$ in the $\mathbf{R}^2$ plane, where the time $t \in \mathbf{R}$. We can introduce[20,21] a space $\mathbf{R} \times \mathbf{R}^2$, the vector field, $(1, \mathbf{V})$ which acts in it and a Lie derivative of the scalar field $F(x, y, t)$ with respect to the vector field $(1, \mathbf{V})$

$$\mathcal{L}_{(1,\mathbf{V})}F \equiv \partial F/\partial t + \mathbf{V}\cdot\nabla F \tag{3.1}$$

In a mechanical interpretation it is identical with the substantive derivative of the scalar quantity F for material particles, the motion of which in the $\mathbf{R}^2$ plane is specified by the vector field $\mathbf{V}$.

As noted in Section 1, we can make the complex-valued vector field $\mathbb{V}(z, \bar{z}, t)$ of the form (1.2), which acts in the complex plane, $\mathbb{C}$ correspond to this vector field. Correspondingly we can introduce the space $\mathbf{R} \times \mathbb{C}$ and the vector field $(1, \mathbb{V})$ acting in it.

Suppose $f(z, t)$ is an analytical function of the complex variable z and the real variable t. We will define the derivative of this function as the scalar field, specified in the space $\mathbf{R} \times \mathbb{C}$, in terms of the vector field $(1, \mathbb{V})$ by means of general formula (3.1)

$$\mathcal{L}_{(1,\mathbb{V})}f \equiv \mathcal{L}_{(1,\mathbf{V})}[\mathrm{Re}f + i\mathrm{Im}f]$$

Taking into account the Cauchy-Riemann relations it can be converted to the form[15]

$$\mathcal{L}_{(1,\mathbb{V})}f \equiv \partial f/\partial t + \mathbb{V}\partial f/\partial z \tag{3.2}$$

Further, as stated in Section 2, suppose ζ is an auxiliary plane in which the region Ω_ζ of unchanged form corresponds to the flow region $\Omega_z(t)$. At each instant of time $t \in T \subset \mathbf{R}$ the conformal mapping of the region $\Omega_z(t)$ into Ω_ζ gives the function $f(z, t)$, while the inverse mapping gives the function $g(\zeta, t)$. Introducing the identical transformation of time $\phi(t) \equiv t$, we obtain the mutually inverse mappings (ϕ, f) and (ϕ, g)

$$(\phi, f)\colon T \times \Omega_z(T) \to T \times \Omega_\zeta, \quad (\phi, g)\colon T \times \Omega_\zeta \to T \times \Omega_z(t)$$

Suppose the vector field $(1, \mathbb{V})$ acts in the region $T \times \Omega_z(t)$. As a result of the mapping (ϕ, f) the vector field $(1, \mathbb{U})$ will correspond to it in the region $T \times \Omega_\zeta$. In the mechanical interpretation, the field $\mathbb{V}(z, \bar{z}, t)$ specifies the motion of material points of the region $\Omega_z(t)$, while the field $\mathbb{U}(\zeta, \bar{\zeta}, t)$ specifies the motion of the transforms of these points in the region Ω_ζ for mapping (ϕ, f).

The relations $(1, \mathbb{U}) = \mathcal{L}_{(1,\mathbb{V})}(\phi, f)$, $(1, \mathbb{V}) = \mathcal{L}_{(1,\mathbb{U})}(\phi, g)$ hold for the Lie derivative.[21] Applying formula (3.2) to them, we obtain

$$\begin{aligned} 1 &\equiv \mathcal{L}_{(1,\mathbb{V})}\phi(t), \quad \mathbb{U}(\zeta, \bar{\zeta}, t) = (\partial f/\partial t + \mathbb{V}\partial f/\partial z)|_{z = g(\zeta, t)} \\ 1 &\equiv \mathcal{L}_{(1,\mathbb{U})}\phi(t), \quad \mathbb{V}(z, \bar{z}, t) = (\partial g/\partial t + \mathbb{U}\partial g/\partial\zeta)|_{\zeta = f(z, t)} \end{aligned} \tag{3.3}$$

4. Expression of conditions (2.10) in terms of a Lie derivative

The vector field $\mathbb{U}_g(\zeta, t) \equiv 0$ essentially corresponds to the conformal motion $\mathbb{V}_g = (z, t)$ generated by the mapping (1.3) in the ζ plane. Substituting it into formulae (3.3), we obtain, in addition to expression (1.5), the new formula

$$\mathbb{V}_g(z,t) = (\partial g/\partial t)|_{\zeta = f(z,t)} = -(\partial f/\partial t)/(\partial f/\partial z) \tag{4.1}$$

We can similarly determine another conformal motion, generated by the mapping (1.4), when z acts as the plane of Lagrange variables. The velocity field $\mathbb{V}_f(z, t) \equiv 0$ corresponds to it in this plane, while the field $\mathbb{U}_f(\zeta, t)$, which by analogy with (4.1) has the form

$$\mathbb{U}_f(\zeta,t) = (\partial f/\partial t)|_{z = g(\zeta,t)} = -(\partial g/\partial t)/(\partial g/\partial \zeta) \tag{4.2}$$

corresponds to it in the ζ plane.

Calculating the partial derivatives of the function $g(\zeta, t)$

$$\begin{aligned}\frac{\partial g}{\partial \zeta} &= \frac{1}{\pi\zeta} - \frac{1}{\zeta^2}\sum_{j=1}^{J}\sum_{k=0}^{K_j}\frac{d_{k,j}}{(\zeta^{-1}-b_j^{-1})^{k+1}} - \sum_{m=1}^{M}\frac{m\beta_m}{\zeta^{m+1}}\\ \frac{\partial g}{\partial t} &= 1 + \sum_{j=1}^{J} b'_j(t)\left[\sum_{k=0}^{K_j}\frac{d_{k,j}}{(\zeta^{-1}-b_j^{-1})^{k+1}b_j^2}\right] - \sum_{j=1}^{J}\sum_{k=1}^{K_j}\frac{d'_{k,j}(t)}{(\zeta^{-1}-b_j^{-1})^k k} + \sum_{m=1}^{M}\frac{\beta'_m(t)}{\zeta^m}\end{aligned} \tag{4.3}$$

we can judge the singularities of the function $\mathbb{U}_f(\zeta, t)$. It is obvious that it is rational in the ζ plane, and has a simple pole at infinity and a pole $a_k(t)$ which lies inside the unit circle, by virtue of requirement (2.4). Consequently, the function $\mathbb{U}_f(\zeta, t)$ is regular everywhere in the region Ω_ζ.

From formulae (1.10) and (4.2) we obtain the following new representation of the function $\Phi(\zeta, t)$

$$\Phi(\zeta,t) \equiv \zeta(\partial g/\partial\zeta)\mathcal{L}_{(1,\mathbb{U}_f)}r(\zeta,t)$$

Then the boundary equation (2.9) can be rewritten in terms of the Lie derivative of the Schwarz function $r(\zeta, t)$

$$\zeta = e^{i\sigma}:\ \zeta\frac{\partial g}{\partial\zeta}\mathcal{L}_{(1,\mathbb{U}_f)}r(\zeta,t) = \frac{2}{\pi}Q(t) \tag{4.4}$$

According to the corollary, this can be satisfied by satisfying conditions (2.10). Since all the poles of the rational function $\zeta\partial g/\partial\zeta$ lie outside $\bar{\Omega}_\zeta$, this is equivalent to the set of conditions

$$\mathcal{L}_{(1,\mathbb{U}_f)}r(\zeta,t)|_{\zeta\sim\bar{b}_j^{-1}} = O(1),\quad j = 1,\ldots,J_0+J_1;\quad \mathcal{L}_{(1,\mathbb{U}_f)}r(\zeta,t)|_{|\zeta|\sim\infty} = O(1) \tag{4.5}$$

Using transformation (1.8) we construct the Schwarz function for a solution of the form (2.5), (2.6), apart from an unimportant constant,

$$r(\zeta,t) = t - \frac{\ln\zeta}{\pi} + \sum_{j=1}^{J}\left[\bar{d}_{j,0}\ln(\zeta-\bar{b}_j^{-1}) - \sum_{k=1}^{K_j}\frac{\bar{d}_{j,k}}{(\zeta-\bar{b}_j^{-1})^k k}\right] + \sum_{m=1}^{M}\beta_m\zeta^m \tag{4.6}$$

Taking (3.2) into account we will analyse the left-hand sides of conditions (4.5) for the presence of singularities in the region of closure $\bar{\Omega}_\zeta$. They can be generated by singularities of the function $\mathbb{U}_f = (\zeta, t)$ and of the partial derivatives of the function $r(\zeta, t)$. As already noted above, the function $\mathbb{U}_f(\zeta, t)$ in the closure $\bar{\Omega}_\zeta$ has a unique singularity – a simple pole at infinity. Further, from expression (4.6) we obtain $\partial r/\partial\zeta$ and $\partial r/\partial t$ directly by differentiation. Both these partial derivatives can be represented in the form of the sum of terms, each of which has a singularity at the point $\bar{b}_j^{-1}$, which it contains explicitly. Correspondingly on the left-hand sides of the j-th local condition (4.5), only those terms of representation (4.6) which explicitly contain $\bar{b}_j^{-1}$ can make an important contribution. As a result, the first group of conditions (4.5) gives

$$\mathcal{L}_{(1,\mathbb{U}_f)}\left[\bar{d}_{j,0}\ln(\zeta-\bar{b}_j^{-1})-\sum_{k=1}^{K_j}\frac{\bar{d}_{j,k}}{(\zeta-\bar{b}_j^{-1})^k k}\right]\Bigg|_{\zeta\sim\bar{b}_j^{-1}} = O(1), \quad j = 1, \dots, J_0+J_1 \tag{4.7}$$

Similarly, on the left-hand side of the last local condition in (4.5), in the neighbourhood of infinity, only the last term of the additive representation (4.6)

$$\mathcal{L}_{(1,\mathbb{U}_f)}\left[\sum_{m=1}^{M}\beta_m\zeta^m\right]\Bigg|_{|\zeta|\sim\infty} = O(1) \tag{4.8}$$

can make an important contribution.

As a result, the set of local conditions (2.10) can be reduced to the simpler set of local conditions (4.7) and (4.8).

5. Formulation of the dynamic problem

The free parameters of the solution can be treated as the phase coordinates of a certain generalized dynamical system. The state of the system at the instant t is represented by the vector $\mathbf{X}(t)$ of the phase space of a system of dimension L

$$\mathbf{X}(t) = \{[\beta_m(t), m = 1, \dots, M], [b_j(t), d_{j,k}(t); j = 1, \dots, J_0+J_1; k = 1, \dots, K_j]\} \tag{5.1}$$

We will show that the imposition of local conditions (4.7), (4.8) leads to a generalized problem of the dynamics of a system of the form

$$\sum_{l=1}^{L}A_{l,s}(\mathbf{X})X_l'(t) = B_s(\mathbf{X}), \quad s = 1, \dots, L \tag{5.2}$$

were $\mathbf{B}(\mathbf{X})$ is a vector of dimension L and $\mathbf{A}(\mathbf{X})$ is a square $L\times L$ matrix, where both objects are functions of only the vector $\mathbf{X}(t)$.

We will write condition (4.7) taking formula (3.2) into account

$$\left[\left(\mathbb{U}_f-\frac{\partial\bar{b}_j^{-1}}{\partial t}\right)\sum_{k=1}^{K_j+1}\frac{\bar{d}_{j,k-1}}{(\zeta-\bar{b}_j^{-1})^k}-\sum_{k=1}^{K_j}\frac{\bar{d}'_{j,k}(t)}{(\zeta-\bar{b}_j^{-1})^k k}\right]\Bigg|_{\zeta\sim\bar{b}_j^{-1}} = O(1), \quad j = 1, \dots, J_0+J_1 \tag{5.3}$$

Similarly, writing condition (4.8), we obtain

$$\left\{\beta'_M(t)\zeta^M+\sum_{m=1}^{M-1}[\beta'_m(t)+(m+1)\beta_{m+1}\mathbb{U}_f]\zeta^m\right\}\Bigg|_{|\zeta|\sim\infty} = O(1) \tag{5.4}$$

In view of the regularity of the function $\mathbb{U}_f(\zeta, t)$ in the region Ω_ζ, its expansion in a Taylor series with coefficients $C_{j,n}(t)$ holds

$$\mathbb{U}_f(\zeta,t)|_{\zeta\sim\bar{b}_j^{-1}} = \sum_{n=0}^{\infty}C_{j,n}(\zeta-\bar{b}_j^{-1})^n, \quad j = 1, \dots, J_0+J_1 \tag{5.5}$$

In the neighbourhood of infinity we have $\mathbb{U}_f(\zeta, t) \sim \zeta$ and correspondingly the following expansion of the function $\mathbb{U}_f(\zeta, t)$ in series with coefficients $C_{\infty,n}(t)$ holds

$$\mathbb{U}_f(\zeta,t)|_{|\zeta|\sim\infty} = \sum_{n=0}^{\infty}\frac{C_{\infty,n}}{\zeta^{n-1}} \tag{5.6}$$

We substitute expansion (5.5) into the j-th condition (5.3). Then only the additive terms of order $(\zeta - \bar{b}_j^{-1})^{-k}(k = 1, \ldots, K_j + 1)$ make an important contribution, i.e. which differs from $O(1)$, to the left-hand side of the condition

$$\left\{\left[\sum_{n=0}^{K_j} C_{j,n}(\zeta - \bar{b}_j^{-1})^n - \frac{\partial \bar{b}_j^{-1}}{\partial t}\right]\sum_{k=1}^{K_j+1} \frac{\bar{d}_{j,k-1}}{(\zeta - \bar{b}_j^{-1})^k} - \sum_{k=1}^{K_j} \frac{\bar{d}'_{j,k}(t)}{(\zeta - \bar{b}_j^{-1})^k k}\right\}\Bigg|_{\zeta \sim \bar{b}_j^{-1}} = O(1) \tag{5.7}$$

In order to obtain the conditions for equality (5.7) to be satisfied, we will group terms of the same order and equate the resulting coefficient of them to zero. Of the number of significant orders $k = 1, \ldots, K_j + 1$ we obtain $K_j + 1$ equations.

We will first write an equation for the leading order $k = K_j + 1$

$$\partial \bar{b}_j^{-1}/\partial t = C_{j,0}(t), \quad j = 1, \ldots, J_0 + J_1 \tag{5.8}$$

Taking this into account, we rewrite condition (5.7) in the simpler form

$$\left[\sum_{k=2}^{K_j+1}\sum_{n=1}^{k-1} \frac{C_{j,n}\bar{d}_{j,k-1}}{(\zeta - \bar{b}_j^{-1})^{k-n}} - \sum_{k=1}^{K_j} \frac{\bar{d}'_{j,k}(t)}{(\zeta - \bar{b}_j^{-1})^k k}\right]\Bigg|_{\zeta \sim \bar{b}_j^{-1}} = O(1)$$

and, grouping terms of order $k = 1, \ldots, K_j$ in it, we obtain

$$\bar{d}'_{j,k}(t) = k\sum_{n=1}^{K_j+1-k} C_{j,n}(t)\bar{d}_{j,n+1-1}(t), \quad j = 1, \ldots, J_0 + J_1, \quad k = 1, \ldots, K_j \tag{5.9}$$

As a result, relations (5.8) and (5.9) give the evolution equations for the set of components $b_j(t)$ and $d_{j,k}(t)$ of the vector $\mathbf{X}(t)$ of phase space. It remains to obtain the evolution equations for the set of components $\beta_m(t)$ of the vector $\mathbf{X}(t)$.

Proceeding in the same way, we substitute expansion (5.6) into condition (5.4). In this case only the additive terms of order $\zeta^k(k = 1, \ldots, M)$ make a considerable contribution to the left-hand side of the condition

$$\left\{\beta'_M(t)\zeta^M + \sum_{m=1}^{M-1}\left[\beta'_m(t)\zeta^m + (m+1)\beta_{m+1}\sum_{n=0}^{m} C_{\infty,n}\zeta^{m-n+1}\right]\right\}\Bigg|_{|\zeta| \sim \infty} = O(1)$$

Grouping terms of the same order m and equating the resulting coefficient for ζ^m to zero, we obtain

$$\beta'_m(t) = -\sum_{n=0}^{M-m} (m+n)C_{\infty,n}(t)\beta_{m+n}(t), \quad m = 1, \ldots, M \tag{5.10}$$

Hence, a closed system of ordinary differential equations (5.8)–(5.10) is obtained for the components of the vector $\mathbf{X}(t)$ of the phase space of system (5.1). To analyse it it is necessary to investigate the structure of the functions $C_{j,n}(t)$, $C_{\infty,n}(t)$ – the coefficients of the expansions (5.5) and (5.6) in a Taylor series of the function $\mathbb{U}_f(\zeta, t)$, represented by expression (4.2).

We will now consider relations (4.3). The partial derivative $\partial g/\partial \zeta$ is a function of the complex variable ζ and the components of the vector $\mathbf{X}(t)$. The partial derivative $\partial g/\partial t$ is also a function of the complex variable ζ of the components of the vector $\mathbf{X}(t)$ and, moreover, of its derivative $\mathbf{X}'(t)$, where the dependence on the vector $\mathbf{X}'(t)$ is obviously linear. Consequently, the function $\mathbb{U}_f(\zeta, t)$), can be represented in the form

$$\mathbb{U}_f(\zeta, t) = U_0(\zeta, \mathbf{X}) + \sum_{l=1}^{L} U_l(\zeta, \mathbf{X})X'_l(t)$$

where $U_l(\zeta, \mathbf{X})$ $(l = 0, \ldots L)$ are certain functions (their actual form is easily obtained from representation (4.2) of the function $\mathbb{U}_f(\zeta, t)$ taking into account expression (4.3) and is not given here in view of its complicated form). Then, the coefficients $C_{j,n}(t)$, $C_{\infty,n}(t)$ of the expansions of the function $\mathbb{U}_f(\zeta, t)$ have the structure

$$C_{\chi,n}(t) = \hat{U}_{0,\chi,n}(\mathbf{X}) + \sum_{l=1}^{L} \hat{U}_{l,\chi,n}(\mathbf{X}) X_l'(t), \quad \chi = j, \infty \tag{5.11}$$

We will analyse the system of equations (5.8)–(5.10) taking expressions (5.11) into account. Obviously, on the right-hand sides of all the equations the dependence on t is not explicit, since the components of the vector $\mathbf{X}(t)$ depend on t, while the dependence on the components of the derivative $\mathbf{X}'(t)$ is linear. Consequently, the whole set of equations (5.8)–(5.10) can be represented in the form (5.2).

6. The law of variation of the flow rate through the channel

We will find to what form of law of variation of the flow rate $Q_*(t)$ in the channel the solution of the problem of the form (2.5), (2.6) corresponds, when its free parameters (5.1) satisfy the generalized problem of the dynamics of system (5.2). To do this we must calculate the left-hand side at an arbitrary point of the ζ plane in boundary equation (2.9). In this way we can obtain a set of expressions for the function $Q_*(t)$, different in form but essentially equivalent. It is not appropriate to derive them here since the most convenient expression will be obtained below when integrating the dynamical system (5.2) (see Section 8).

Suppose the function $Q_*(t)$ has been obtained. We will introduce a new time θ,[19] related to the old time t by the one-to-one correspondence $\theta = \theta(t)$, so that $\theta'(t) > 0$ for all $t > 0$ and $\theta(t) = 0$ when $t = 0$. We will determine the new mapping function $z = \tilde{g}(\zeta, \theta)$ and the Schwarz function $\tilde{r}(\zeta, \theta)$ corresponding to it so that

$$\tilde{g}(\zeta, \theta) \equiv g(\zeta, t)|_{t = t(\theta)}, \quad \tilde{r}(\zeta, \theta) \equiv r(\zeta, t)|_{t = t(\theta)} \tag{6.1}$$

Substituting the first expression of (1.10) for the function $\Phi(\zeta, t)$ into Eq. (2.9) and multiplying both sides by $t'(\theta)$, we obtain an equation of the structure, analogous to (2.9), only for the time θ and a law of variation of the flow rate through the channel of the form $Q(\theta) = t'(\theta) Q_*[t(\theta)]$. Integrating over θ we obtain the relation

$$\int_0^t Q_*(t)dt = \int_0^\theta Q(\theta)d\theta \tag{6.2}$$

which enables us to establish a unique correspondence between θ and t for any prespecified law of variation of the flow rate $Q(\theta) > 0$ so long as the condition $Q_*(t) > 0$ is satisfied. Hence, from the solution of the Hele–Shaw problem of the form (2.5) for a definite law of variation of the flow rate in the channel $Q_*(t)$ and using substitution (6.1) we can obtain a solution of the problem for any prespecified law $Q(\theta) > 0$, thereby establishing the correspondence $t(\theta)$ using formula (6.2).

As a rule, for flows in a channel the flow rate remains fixed: $Q(\theta) = 1$. Hence, when $Q_*(t) > 0$ it immediately follows from expression (6.2) that

$$\theta(t) = \int_0^t Q_*(t)dt \tag{6.3}$$

7. Comparison with the original Howison method

Using the mapping (ϕ, g) (see Section 3) we will transfer from the region $T \times \Omega_\zeta$ to the region $T \times \Omega_z(t)$. Taking the invariance of the Lie derivative under the mappings into account[21] and determining the functions $r(\zeta, t)$ we obtain

$$z \in \Gamma(t): \{\mathcal{L}_{(1, \mathbb{U}_f)} r(\zeta, t)\}_{\zeta = f(z,t)} = \mathcal{L}_{(1, \mathbb{V}_f)} S(z, t) \equiv \partial S/\partial t$$

Then, dividing Eq. (4.4) by $\zeta(\partial g/\partial \zeta)$ and taking relations (2.11) into account, we obtain one more form of the boundary evolution equation

$$z \in \Gamma(t): \partial S/\partial t = 2\partial W/\partial z \tag{7.1}$$

It was precisely this form of the boundary evolution equation that was used by Howison.[10] In this case the unit circle with a cut was chosen, not entirely successfully, as the auxiliary plane. In order that the partial derivatives of the function $g(\zeta, t)$ should be rational functions in the ζ plane, it is necessary to join it along the cut. This can only be done if the solution is symmetrical about the central axis of the Hele–Shaw channel. The class of solutions is thereby artificially narrowed down.

Boundary equation (7.1) was later extended to the closure of the flow region $\bar{\Omega}_z(t)$. The derivative of the complex flow potential $\partial W/\partial z$ in $\bar{\Omega}_z(t)$ is regular. To satisfy Eq. (7.1) everywhere in $\bar{\Omega}_z(t)$, all the singularities of the derivative of the Schwarz function $\partial S/\partial t$ must vanish there. This gives a number of conditions, imposed on the free parameters of the solution $g(\zeta, t)$, where these conditions are algebraic, i.e. they are actually first integrals of the generalized problem of the dynamics of the system considered in Section 6. At the same time, from the character of the derivation, they are necessary but not sufficient conditions for satisfying the evolution equation (7.1), since the singularities of the functions $\partial S/\partial t$ and $\partial W/\partial z$ are only defined in the region $\Omega_z(t)$. Without having representations on the singularities of the functions $\partial S/\partial t$ and $\partial W/\partial z$ in the remaining part of the z plane, we cannot judge the equality between them either in the closure of the region $\bar{\Omega}_z(t)$ or, in particular, at the interface $\Gamma(t)$.

Hence, Howison's method in the original does not enable us justifiably to conclude whether the parametrized form of the mapping function $g(\zeta, t)$ is a solution of the Hele–Shaw problem (1.1) or not. At the same time, if this fact is established from some additional considerations (as, for example, here), Howison's method can be used to integrate a dynamic problem of the type (5.2). We will show this.

8. Integration of the dynamical system (5.2)

Suppose the free parameters satisfy problem (5.2). Then Eq. (4.4) is satisfied over the whole ζ plane, while Eq. (7.1) is satisfied in the closure of the region $\bar{\Omega}_z(t)$. Taking into account the regularity of the function $\bar{\Omega}_z(t)$ in $\partial W/\partial z$, we have the relation

$$z \in \bar{\Omega}_z(t)\colon \partial S/\partial t = O(1) \tag{8.1}$$

The singularities of the function $r(\zeta, t)$ are infinity and the points $\zeta = \bar{b}_j^{-1}(t)$. Correspondingly, the singularities of the function $S(z, t)$ are infinity and the points $z = B_j(t)$, and the transforms of the points $\zeta = \bar{b}_j^{-1}(t)$ for the mapping $z = g(\zeta, t)$

$$B_j(t) = g(\zeta, t)|_{\zeta = \bar{b}_j^{-1}(t)} \tag{8.2}$$

We will analyse the behaviour of the function $S(z, t)$ in the neighbourhood of the points $B_j(t)$. In view of the regularity of the function $g(\zeta, t)$ at the point $\zeta = \bar{b}_j^{-1}(t)$, we can write the Burmann-Lagrange series[16] (the prime denotes a derivative with respect to ζ)

$$\zeta - \bar{b}_j^{-1} = \sum_{n=1}^{\infty} A_{j,n}(t)(z - B_j)^n, \quad A_{j,1}(t) = \left.\frac{1}{g'}\right|_{\zeta = \bar{b}_j^{-1}}, \quad A_{j,2}(t) = -\left.\frac{g''}{2(g')^3}\right|_{\zeta = \bar{b}_j^{-1}}, \ \ldots \tag{8.3}$$

From representation (4.6) we then find the principal part of the Laurent expansion of the combination $S(z, t) - \bar{d}_{j,0}\ln[z - B_j(t)]$ in the neighbourhood of the point $B_j(t)$

$$S(z, t)|_{z \sim B_j} = \bar{d}_{j,0}\ln[z - B_j(t)] + \sum_{k=1}^{K_j} D_{j,k}(t)[z - B_j(t)]^{-k} + O(1) \tag{8.4}$$

The coefficients of the expansion of $D_{j,k}(t)$ can be expressed in terms of the set of coefficients $\{\bar{d}_{j,k}(t), \quad A_{j,k}(t), \quad k = 1, \ldots, K_j\}$. In particular, we present the form of the coefficient for the leading term of the expansion

$$D_{j,K_j}(t) = -K_j^{-1}\bar{d}_{j,K_j}(t)A_{j,1}^{-K_j}(t) \tag{8.5}$$

It follows from representation (8.4) of the Schwarz function $S(z, t)$ that relation (8.1) in the neighbourhood of the point $B_j(t)$ can only be satisfied when the following conservation-law type conditions are satisfied

$$B_j(t) = B_j(0), \quad D_{j,k}(t) = D_{j,k}(0); \quad j = 1, \dots, J_0 + J_1; \quad k = 1, \dots, K_j \tag{8.6}$$

The corresponding conservation laws for $j \geq J_0 + J_1 + 1$ will obviously be satisfied automatically by virtue of subdivision (2.6).

We can similarly obtain the laws of conservation from an analysis of the behaviour of the Schwarz function $S(z, t)$ in the neighbourhood of infinity. The main difference is the fact that infinity is not a point of regularity of the mapping function $z = g(\zeta, t)$. Hence, we will first introduce the auxiliary variables ω and u: $\omega = \zeta^{-1}$, $u = e^{-\pi z}$ and, taking into account the form (2.5) of the function $g(\zeta, t)$, we obtain the relation

$$u(\omega, t) = \omega e^{-\pi\Psi}, \quad \Psi(\omega, t) = t + \sum_{j=1}^{J}\left[d_{j,0}\ln(\omega - b_j^{-1}) - \sum_{k=1}^{K_j}\frac{d_{j,k}}{(\omega - b_j^{-1})^k k}\right] + \sum_{m=1}^{M}\beta_m\omega^m \tag{8.7}$$

The point of regularity of the function $u(\omega, t)$, i.e. $\omega = 0$, corresponds to infinity in the ζ plane. We can therefore write the Burmann-Lagrange series[16]

$$\omega(u, t) = \sum_{n=1}^{\infty}A_{\infty,n}(t)u^n, \quad A_{\infty,1}(t) = e^{\pi\Psi(0,t)}, \quad A_{\infty,2}(t) = e^{\pi\Psi(0,t)}, \dots$$

and, changing to the variable z in expression (4.6), we obtain the estimate

$$S(z, t)|_{\mathrm{Re}z\sim\infty} = t + \left(1 - \pi\sum_{j=1}^{J}\bar{d}_{j,0}\right)[\Psi(0, t) - z] + R \tag{8.8}$$

where

$$R = \sum_{m=1}^{M}\frac{\beta_m(t)}{\omega^m}\bigg|_{\omega=\omega(u,t)} + o(1) = \sum_{m=1}^{M}\beta_m(t)u^{-m}\left[\sum_{n=0}^{M}A_{\infty,n+1}(t)u^n\right]^{-m} + o(1) \tag{8.9}$$

Since the point $u = 0$ corresponds to infinity $\mathrm{Re}z \to \infty$, we can obtain the principal part of the Laurent expansion of the function $R(u, t)$ in the neighbourhood of this point. Making the substitution $u = e^{-\pi z}$, we obtain

$$R = \sum_{m=1}^{M}D_{\infty,m}(t)e^{\pi mz} + D_{\infty,0}(t) + o(1), \quad D_{\infty,M}(t) = \beta_M(t)e^{-\pi\Psi(0,t)}, \dots \tag{8.10}$$

The coefficients $\{D_{\infty,k}(t), k = 1, \dots, M\}$ of the principal part of the expansion are expressed in terms of the set of coefficients $\{\beta_m(t), A_{\infty,m}(t), m = 1, \dots, M\}$.

It follows from representation (8.8) and (8.10) of the Schwarz function $S(z, t)$ that relation (8.1) in the neighbourhood of infinity can only be satisfied when the following conservation-law type conditions are satisfied

$$D_{\infty,m}(t) = D_{\infty,m}(0), \quad m = 1, \dots, M \tag{8.11}$$

The total number of conservation laws (8.6) and (8.11) is equal to the number of free parameters (5.1). Consequently, the dynamic problem is completely integrated. Using expressions (8.2), (8.3), (8.5) and (8.10) we can easily write an explicit form of some first integrals.

The coefficients $D_{\infty,0}(t)$ of expansion (8.10) are necessary in order to obtain the relation $\theta(t)$, by means of which we can construct the solution of the problem for a flow rate in the channel $Q(\theta) = 1$ (see Section 6). In fact, the obvious estimate $\partial W/\partial z = Q_*(t) + o(1)$ at infinity enables us to obtain an explicit form of the term of the order of unity on the right-hand side of relation (8.1). The corresponding terms on the left-hand side can be found from estimate (8.8), (8.10). As a result we obtain

$$2Q_*(t) = 1+\left(1-\pi\sum_{j=1}^{J}\bar{d}_{j,0}\right)\Psi'(0,t)+D'_{\infty,0}(t)$$

and, using relations (6.3), we obtain

$$2\theta(t) = t+\left(1-\pi\sum_{j=1}^{J}\bar{d}_{j,0}\right)[\Psi(0,t)-\Psi(0,0)]+[D_{\infty,0}(t)-D_{\infty,0}(0)] \tag{8.12}$$

The coefficient $D_{\infty,0}(t)$, like the remaining coefficients $D_{\infty,k}(t)$ can be expressed in terms of the set of coefficients $\beta_m(t)$, $A_{\infty,m}(t)$, but the expression obtained is cumbersome. Taking into account the usefulness of formula (8.12), we can indicate an alternative method of calculating the coefficient $D_{\infty,0}(t)$. It has the same meaning as the residue of the function $u^{-1}R(u, t)$ at the point[16] $u=0$

$$D_{\infty,0}(t) = \operatorname*{res}_{0}[u^{-1}R(u,t)] \equiv \frac{1}{2\pi i}\oint_{\gamma_u} u^{-1}R(u,t)du$$

where γ_u is a circle of sufficiently small radius with centre at the point $u=0$. Since $\omega=0$ – the point of regularity of the function $u(\omega, t)$, corresponds to this point, we can change from integration with respect to u to integration with respect to ω

$$D_{\infty,0}(t) = \frac{1}{2\pi i}\oint_{\gamma_\omega} P(\omega,t)d\omega \equiv \operatorname*{res}_{0}P(\omega,t), \quad P(\omega,t) = R(u,t)\frac{\partial \ln u}{\partial\omega}, \quad u = u(\omega,t)$$

where γ_ω is a circle with centre at the point $\omega=0$, the transform of the circle γ_u for the mapping $\omega(u, t)$. Substituting expressions (8.7) and (8.9) here we obtain

$$P(\omega,t) = \left[\sum_{m=1}^{M}\frac{\beta_m(t)}{\omega^m}+o(1)\right]\left[\frac{1}{\omega}-\sum_{k=1}^{J}\sum_{k=1}^{K_j+1}\frac{\pi d_{j,k-1}}{(\omega-b_j^{-1})^k}-\pi\sum_{m=1}^{M}m\beta_m\omega^{m-1}\right] \tag{8.13}$$

The term containing the double sum, in the neighbourhood of the point $\omega=0$, can be expanded in a Taylor series with coefficients[22]

$$E_n(t) = \pi\sum_{j=1}^{J}\sum_{k=1}^{K_j+1}(-1)^{k+n}\binom{-k}{n}d_{j,k-1}(t)b_j^{k+n}(t), \quad n\geq 0$$

Then the coefficient $D_{\infty,0}(t)$ is found directly from expression (8.13) as the residue of the function $P(\omega, t)$ at the point $\omega=0$

$$D_{\infty,0} = -\sum_{m=1}^{M}\beta_m(t)[E_{m-1}(t)+\pi m\beta_m(t)]$$

9. Discussion of the results

Note that the first group of conservation laws in expression (8.6) also follows directly from the dynamic equations (5.8). Actually, taking into account the fact that the first coefficient has the form $C_{j,0}(t) = \mathbb{U}_f(\zeta, t)$, where $\zeta = \bar{b}_j^{-1}(t)$, we can convert Eq. (5.8) to the form

$$\partial\bar{b}_j^{-1}/\partial t = \mathbb{U}_f(\zeta,t)|_{\zeta=\bar{b}_j^{-1}(t)}, \quad j = 1, \dots, J_0+J_1$$

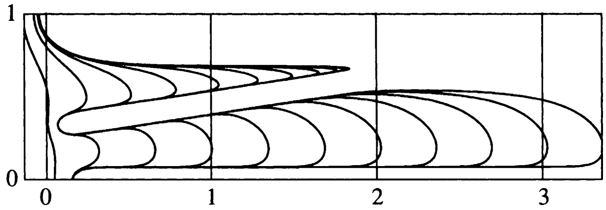

Fig. 2.

This means that the points $\bar{b}_j^{-1}(t)$ of the ζ plane are transformed by the vector field $\mathbb{U}_f(\zeta, t)$. In accordance with the mechanical analogy in Section 3, the mapping (ϕ, g) transfers the point $\bar{b}_j^{-1}(t)$ of the ζ plane into their images $B_j(t)$ in the z plane (see formula (8.2)). The motion of the images is described by the vector field $\mathbb{V}_f(z, t) \equiv 0$, i.e. the position of the points $B_j(t)$ in the z plane does not change,[11] which, in fact, also expresses the first group of conservation laws (8.6).

The remaining conservation laws in expressions (8.6) and (8.11) do not have the same clear mechanical interpretation, which emphasises the significance of Howison's method of integrating dynamical system (5.2).

Note the nature of condition (2.4). It needs to be satisfied only at the initial instant of time by choosing the values of the free parameters (5.1), so that all the singularities of the mapping function $g(\zeta,0)$ lie inside the unit circle. As time passes the position of the singularities changes, but, by continuity, condition (2.4), is satisfied, at least, for a certain time. At some instant both the pole $B_j(t)$ and the zero $a_n(t)$ of the derivative $\partial g/\partial\zeta$ may approach the boundary of the unit circle. In the first case the situation is not critical: the pole approaches the interface $\Gamma(t)$, but never reaches it.[10] A slit is then formed on $\Gamma(t)$ which separates two fingers (finger tip splitting). In the second case the situation is critical: a cusp is formed at the interface, the velocity at this point becomes infinite, and the classical solution of the problem ceases to exist[19] (an example is the Polubarinova-Kochina solution[5] for a cardioid).

Hence, we have constructed an exact solution of the problem of fingering in a Hele–Shaw channel of general form from the class of parametrized solutions, for which the partial derivatives of the mapping function are rational in the auxiliary plane. It contains all known exact solutions,[10] but, unlike them, allows of asymmetry about the channel axis and freedom of choice of the order of the poles – the singularities of the mapping function in the auxiliary plane. For free parameters of the solution we have written a generalized problem of the dynamics of the system and we have shown that it is completely integrable. The latter fact does not reduce the significance of the dynamic formulation of the problem, since it is best to use it for specific calculations, while the conservation laws, being essentially non-linear algebraic equations, are invoked to estimate the accuracy of the integration.

In Fig. 2 we show an example of the calculation of the asymmetrical fingering in a channel, similar to that obtained previously in Ref. 23 in the asymptotic limit as $t \to \infty$ when the initial values of the free parameters were chosen as follows:

$$b_1(0) = -b_2(0) = 0.6, \quad b_3(0) = 0.3 + i0.7, \quad b_4(0) = \bar{b}_3(0)$$

$$a_1(0) = a_2(0) = 0.2, \quad a_3(0) = 0.2143 + i0.5, \quad a_4(0) = \bar{a}_3(0)$$

Then, the only non-zero values of the parameters $d_{j,k}(t)$ will be constant:

$$d_{1,0} = 0.04863, \quad d_{2,0} = 0.1987, \quad d_{3,0} = 0.0445 + i0.006714, \quad d_{4,0} = \bar{d}_{3,0}$$

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